Multiple-scale perturbation analysis of the direct interaction approximation for inertial-range turbulence

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A technique from multiple-scale singular perturbation theory is used to derive a solution to the equations of the direct interaction approximation for turbulence valid in the limit $\varepsilon \to 0$. This approach permits non-self-similar solutions to be derived which reduce to the renormalization-group results in the special case of self-similar forcing.

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There has been a fair amount of published work in recent years exploring the potential for a renormalization-group analysis of fluid mechanical turbulence (for example, Refs. [1–5]). The one conclusion that may safely be drawn from this work is that if such an approach is to yield a satisfactory theory of turbulence, a more thorough understanding of its nature is necessary. In this spirit, the present paper clarifies the relationship between the renormalization group as applied to turbulence and the multiple-scale perturbation techniques of singular perturbation theory. In the process, we extend the range of accessible problems to those with certain types of non-self-similar forcing. It is hoped that the technique proposed here might also have applications to non-self-similar versions of problems from other areas of physics which have been solved by renormalization-group techniques.

Each of the available techniques for deriving inertial-range solutions for turbulence (the various versions of the renormalization group, the similarity-solution–asymptotic-expansion technique of [4], etc.) involves two types of approximations: statistical approximations for closing the infinite heirarchy of moment equations and whatever additional approximations are necessary to solve the equations that result from the statistical closure approximations. It has been noted [4,6] that the statistical closure approximation underlying the renormalization-group analysis of turbulence is equivalent to the direct interaction approximation (DIA) of Kraichnan [7]. The renormalization-group techniques we are concerned with here employ the ε expansion as the additional approximation to make the problem analytically tractable. That the DIA plus the ε expansion are sufficient to yield the Yakhot-Orszag renormalization-group results [2] was shown explicitly in Refs. [4,8], where a similarity solution for the DIA integral equations was presented which yielded the Yakhot-Orszag results when expanded for small ε.

The purpose of the present paper is to show how an ε-expansion solution to the DIA equations may be derived using singular perturbation theory. This technique makes no use of the self-similarity inherent in the renormalization-group analysis and so may potentially be used to attack a much wider class of problems. In the present work, the response of the inertial range to one type of non-self-similar forcing is examined. (By inertial range is meant the range of length and time scales between the smallest viscous dissipation scales and the largest energy-containing scales. In this work, the inertial range is dominated by inertial effects, but it is not necessarily self-simila-

We are concerned with the statistics of solutions to the Navier-Stokes equations,

$$\frac{\partial u_i}{\partial x_i}=0 \ , \qquad \left[\frac{\partial}{\partial t}-\nu\frac{\partial^2}{\partial x_i \partial x_i}\right]u_i+\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}(u_i u_j)=f_i \ , \tag{1}$$

when forced by the Gaussian random vector field $f_i(\mathbf{x},t)$ with the two-point correlation

$$\langle f_i(t,\mathbf{k}) f_j(t,\mathbf{k})\rangle=\frac{1}{4\pi k^2}F(t-t',k)P_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}') \ . \tag{2}$$

We shall consider the special case of forcing with $F(t-t',k)=F_0(k)\delta(t-t')$.

The DIA provides equations for the infinitesimal response function and the two-point correlation function describing the statistics of the solutions of these equations. For the statistically homogeneous, stationary and isotropic turbulence studied here, these tensors may be written, in the time–wave-number domain,

$$\left\langle \frac{\delta u_i(\mathbf{k},t)}{\delta f_j(\mathbf{k}',t')}\right\rangle = G(t-t',k)P_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}') \ , \qquad \langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t')\rangle=\frac{1}{4\pi k^2}E(t-t',k)P_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}') \ , \tag{3}$$

where $P_{ij}(\mathbf{k})=\delta_{ij}-k_i k_j/k^2$. The functions $G(t,k)$ and $E(t,k)$ satisfy the equations [9]

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]G(t,k)+\frac{1}{2}k^2\int_0^\infty dp\int_{-1}^{+1}dz\, b_1\left[\frac{k}{p},z\right]\int_{-\infty}^{+\infty}ds\, G(s,k)G(t-s,q)E(t-s,p)=\delta(t) \ ,$$

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]E(t,k)+\frac{1}{2}k^2\int_0^\infty dp\int_{-1}^{+1}dz\, b_1\left[\frac{k}{p},z\right]\int_{-\infty}^{+\infty}ds\, E(s,k)G(t-s,q)E(t-s,p)$$
$$=\frac{1}{2}k^2\int_0^\infty dp\int_{-1}^{+1}dz\frac{k^2}{q^2}a_1\left[\frac{k}{p},z\right]\int_{-\infty}^{+\infty}ds\, G(s,k)E(t+s,q)E(t+s,p)+\int_{-\infty}^{+\infty}ds\, G(s,k)F(t+s,k)\ , \quad (4)$$

where the geometric coefficients a_1 and b_1 are

$$a_1(x,z)=\frac{(1-z^2)(1+2z^2-3zx+x^2)}{2(1-2xz+x^2)}\ ,$$
$$b_1(x,z)=\frac{(1-z^2)(z-2zx^2+x^3)}{x(1-2xz+x^2)}\ , \quad (5)$$

and q is related to k and p by $q^2=k^2+p^2-2kpz$. The wave-number integrals have been written in a form following Ref. [4] for convenience in the ensuing analysis.

The initial condition for the Green's function is $G(t,k)\to 1$ as $t\to 0+$. At $t=0$, $E(t,k)$ takes the value $E(k)$, the energy spectrum function. It should be remembered that here, since t is the time difference $t-t'$, the "initial conditions" are really conditions on the equal-time values of the two-point functions.

Before proceeding with the main analysis of this paper, it is useful to recall that Kraichnan [3] has identified the fundamental approximation of the ε-expansion renormalization group as a scale-separation assumption, in which it is assumed that the dominant nonlinear interactions affecting a given wave-number $\mathbf{k}$ are those involving wave-number triads $\mathbf{k},\mathbf{p},\mathbf{q}$ with $p=|\mathbf{p}|$ and $q=|\mathbf{q}|$ much larger than $k=|\mathbf{k}|$. Kraichnan implements this assumption directly, with his distant-interaction approximation, by including only interactions with $p,q\gg k$. Kraichnan also pointed out that his result could be obtained perturbatively. The purpose of the present work is to show how singular perturbation theory may be employed to explicitly make use of the multiple-scale nature of the problem in the limit $\varepsilon\to 0$ to derive non-self-similar solutions.

In keeping with our intention of making a multiple-scale, singular-perturbation analysis, we assume that the forcing has the form $F_0(k)=g(k^\varepsilon)k^3$ (defining the small parameter ε), where $g(x)$ is an arbitrary function of its argument. This function is slowly varying in the sense that $k[\partial g(k^\varepsilon)/\partial k]/g(k^\varepsilon)$ is small, just as, more commonly, a function $f(\varepsilon x)$ is considered to be slowly varying because $[\partial f(\varepsilon x)/\partial x]/f(\varepsilon x)$ is small. In a manner similar to that in which weakly nonlinear expansions of the solutions to nonlinear partial differential equations are constructed, the power-law part of $F_0(k)$ is chosen so as to be just on the verge of being strong enough (that is, the random forcing is just long-range enough) to make the nonlinear interactions of the fluctuations important. Similar reasoning leads to making the definition $E(k)=\varepsilon e(k^\varepsilon)k$. Thus, the ε expansion is an approximation based on the smallness of the nonlinear effects, a smallness which is due to the assumed relatively short-range nature of the force correlations.

The perturbation analysis begins with the ansatz

$$G(t,k)=e^{-\nu(k)k^2t}H(t)\ ,\quad E(t,k)=E(k)e^{-\nu(k)k^2|t|}\ . \quad (6)$$

We wish to determine $\nu(k)$ and $E(k)$ in the limit $\varepsilon\to 0$. Substituting the expressions (6) into the DIA Eq. (4) and neglecting exponentials $e^{-\nu(p)p^2t}$ and $e^{-\nu(q)q^2t}$ relative to $e^{-\nu(k)k^2t}$ (since we expect that $p,q\gg k$ in the limit $\varepsilon\to 0$ and $\nu(k)k^2$ is an increasing function of k; this must be verified *a posteriori*) leads to the integral equations

$$\nu(k)=\tfrac{1}{2}\int_0^\infty dp\int_{-1}^{+1}dz\, b_1\left[\frac{k}{p},z\right]\times\frac{E(p)}{-\nu(k)k^2+\nu(p)p^2+\nu(q)q^2} \quad (7)$$

and

$$\nu(k)E(k)+\tfrac{1}{2}\int_0^\infty dp\int_{-1}^{+1}dz\, b_1\left[\frac{k}{p},z\right]\frac{E(k)E(p)}{\nu(k)k^2+\nu(p)p^2+\nu(q)q^2}$$
$$=\tfrac{1}{2}\int_0^\infty dp\int_{-1}^{+1}dz\frac{k^2}{q^2}a_1\left[\frac{k}{p},z\right]E(p)E(q)\frac{2[\nu(p)p^2+\nu(q)q^2]}{[\nu(k)k^2+\nu(p)p^2+\nu(q)q^2]^2}+F_0(k)\ . \quad (8)$$

We begin with (7) and solve it for $\nu(k)$ in terms of $E(k)=\varepsilon e(k^\varepsilon)k$; the function $e(x)$ is as yet unknown. This integral equation represents a singular-perturbation problem in the sense that a straightforward expansion for small ε leads to divergent integrals and useless results. Multiple-scale perturbation techniques are not commonly applied to integral equations; the invariance-condition technique proposed in [10] is used here. This technique has been applied to ordinary [10,11] and partial [12,13] differential equations. The basis of this technique is the idea that a straightforward expansion is valid in a small domain and may be used to construct a solution valid in a larger domain. In the present context, the straightforward expansion is invalid because the integral term, involving an integral over a semi-infinite domain, is larger (by a factor of $1/\varepsilon$) than a straightforward expansion would indicate.

The perturbation technique employed here does not work on Fredholm integral equations such as (7), so we replace the lower limit of the p integral by k and solve the resulting Volterra equation instead. Because the dominant contribution to the integral comes from the region near the upper limit, it is to be expected that the solution to the Volterra equation approaches the solution of the Fredholm equation as $k \to 0$.

The larger domain over which we desire a solution, over which the small-scale expansion is not valid, is parametrized by $\bar{k}$ and we define a new variable $\tilde{k}$ by $k=\bar{k}^{1/\varepsilon}\tilde{k}$. The expansion may then be constructed by splitting the integration domain into a large part and a small part and treating the integral over the small part as small compared to the integral over the large part. Equation (7) then becomes

$$\nu(k)=\frac{1}{2}\left[\int_{\bar{k}^{1/\varepsilon}}^{\infty}dp+\int_{\bar{k}^{1/\varepsilon}\tilde{k}}^{\bar{k}^{1/\varepsilon}}dp\right]\int_{-1}^{+1}dz\, b_1\left[\frac{\bar{k}^{1/\varepsilon}\tilde{k}}{p},z\right]\frac{\varepsilon e(p^{\varepsilon})p}{-\nu(k)\bar{k}^{2/\varepsilon}\tilde{k}^2+\nu(p)p^2+\nu(q)q^2} . \quad (9)$$

Solving for $\nu(k)$ as an expansion in ε by iteration yields

$$\nu(k)=A+\frac{1}{2}\frac{\varepsilon}{A}e(\bar{k})\int_{\tilde{k}}^{1}d\tilde{p}\int_{-1}^{+1}dz\, b_1\left[\frac{\tilde{k}}{\tilde{p}},z\right]\times\frac{\tilde{p}}{-\tilde{k}^2+\tilde{p}^2+\tilde{q}^2}+O(\varepsilon^2) , \quad (10)$$

where

$$A=\frac{\varepsilon}{2}\int_{\bar{k}^{1/\varepsilon}}^{\infty}dp\int_{-1}^{+1}dz\, b_1\left[\frac{k}{p},z\right]\times\frac{e(p^{\varepsilon})p}{-\nu(k)k^2+\nu(p)p^2+\nu(q)q^2} \quad (11)$$

is $O(1)$, per the discussion above, and is a function of $\bar{k}$ and $k=\bar{k}^{1/\varepsilon}\tilde{k}$. New integration variables have been defined by $p=\bar{k}^{1/\varepsilon}\tilde{p}$ and $q=\bar{k}^{1/\varepsilon}\tilde{q}$.

The property of this expansion that prevents it from being useful is that the $O(\varepsilon)$ term becomes large like $\ln\tilde{k}$ as $\tilde{k}\to 0$. Thus, the asymptotic expansion breaks down when $\ln\tilde{k}\sim 1/\varepsilon$ or $\ln(\tilde{k}^{\varepsilon})\sim 1$. The singularity in the integrand which leads to this $\ln\tilde{k}$ behavior may be isolated, allowing us to write

$$\begin{aligned}\int_{\tilde{k}}^{1}d\tilde{p}\int_{-1}^{+1}dz\, b_1\left[\frac{\tilde{k}}{\tilde{p}},z\right]\frac{\tilde{p}}{-\tilde{k}^2+\tilde{p}^2+\tilde{q}^2} &=\int_{\tilde{k}}^{1}d\tilde{p}\int_{-1}^{+1}dz\left[(z-z^3)\frac{\tilde{p}}{\tilde{k}}+2z^2(1-z^2)\right]\frac{1}{2\tilde{p}\left[1-\frac{\tilde{k}}{\tilde{p}}z\right]}+\mathcal{T}_{\text{nonsingular}}\\ &=-\tfrac{2}{5}\ln\tilde{k}+\mathcal{T}_{\text{nonsingular}} ,\end{aligned} \quad (12)$$

where $\mathcal{T}_{\text{nonsingular}}$ represents nonsingular terms. Then the expansion for $\nu(k)$ is

$$\nu(k)=A-\frac{1}{5}\frac{\varepsilon}{A}e(\bar{k})\ln\tilde{k}+\mathcal{T}_{\text{nonsingular}}+O(\varepsilon^2) . \quad (13)$$

The function A is determined by noting that the expansion (13) must be invariant, through $O(\varepsilon)$, under the transformation $\bar{k}'=a^{\varepsilon}\bar{k}$, $\tilde{k}'=a\tilde{k}$, where a is a parameter [10]. The infinitesimal operator for this transformation group is $\varepsilon\bar{k}(\partial/\partial\bar{k})-\tilde{k}(\partial/\partial\tilde{k})$; we may enforce the invariance condition by requiring that

$$\left[\varepsilon\bar{k}\frac{\partial}{\partial\bar{k}}-\tilde{k}\frac{\partial}{\partial\tilde{k}}\right](\text{expansion})=O(\varepsilon^2) . \quad (14)$$

This condition leads to a differential equation for A:

$$\varepsilon\bar{k}A'+\tfrac{\varepsilon}{5}e(\bar{k})A^{-1}=0 . \quad (15)$$

A' represents the derivative of A with respect to $\bar{k}$ with $k=\bar{k}^{1/\varepsilon}\tilde{k}$ held constant; $\bar{k}^{1/\varepsilon}\tilde{k}$ is transparent to the invariance-condition operator. Equation (15) is easily solved to give

$$\nu(k)=A=\left[C+\frac{2}{5}\int_{k^{\varepsilon}}^{\infty}e(s)\frac{ds}{s}\right]^{1/2} , \quad (16)$$

where we have reverted to the original variable k and introduced the constant of integration C.

Inasmuch as we are interested in the limit $k\to 0$, for which the solution to the Volterra integral equation derived here becomes identical to the solution of the original Fredholm integral equation, we assume the integral dominates the constant in this limit and find

$$\nu(k)\sim\left[\frac{2}{5}\right]^{1/2}\left[\int_{k^{\varepsilon}}^{\infty}e(s)\frac{ds}{s}\right]^{1/2} . \quad (17)$$

We next consider the integral equation of (8) for $E(k)$. The integral on the right-hand side of this equation, involving a_1, may easily be shown not to yield the type of logarithmically growing term encountered in the determination of $\nu(k)$, and so this integral does not contribute to the solution for $E(k)$ to the present order of approximation. Consideration of the integral on the left-hand side of (8), and the similar integral in the analysis of (7), reveals that the difference in sign in the denominator of the integrands does not affect the logarithmically divergent term, and so the integrals are equivalent in so far as the present approximation is concerned. Equation (8) may thus be written $2\nu(k)k^2E(k)=F_0(k)$, and we find

$$e(k^{\varepsilon})=\left[\frac{5}{2}\right]^{1/2}\frac{g(k^{\varepsilon})}{\left[\int_{k^{\varepsilon}}^{\infty}e(s)\frac{ds}{s}\right]^{1/2}} , \quad (18)$$

fixing $e(k^{\varepsilon})$ implicitly. Equations (6), (17), and (18) comprise a solution for $G(t,k)$ and $E(t,k)$ valid in the limit $\varepsilon\to 0$ for forcing of the form $F_0(k)=g(k^{\varepsilon})k^3$.

The self-similar inertial-range solution of Refs. [2,4] corresponds to $e(s)=E_0/\varepsilon s^{2/3}$, so that the energy-spectrum function has the power-law form $E(k)=E_0k^{1-2\varepsilon/3}$. Equation (17) then gives the solution $\nu(k)\sim(3E_0/5\varepsilon)^{1/2}k^{-\varepsilon/3}$. If a constant ν_0 is defined as the coefficient of the power of k in $\nu(k)$, then $E_0/2\nu_0^2=5\varepsilon/6$, in agreement with the solution presented in Ref. [4]. As described in that paper, an additional constraint may be imposed which requires $\varepsilon=4$ and provides a second relation between E_0 and ν_0, fixing the Kolmogorov constant.

A singular-perturbation technique has been presented for the solution of the equations of the direct interaction approximation for turbulence which does not require that the solution have a power-law or self-similar form. It was shown that this technique recovers the renormalization-group results in the special case of a self-similar solution, illustrating the multiple-scale, singular-perturbative nature of the renormalization-group approach. The relationship between the renormalization group and singular perturbation theory has also been explored by Goldenfeld and co-workers (see, for example, Ref. [14]). The technique presented here could be used to derive non-self-similar solutions to statistical problems in other areas of physics by applying it to the appropriate integral equations analogous to the DIA equations.

Note added in proof. V. M. Canuto and M. S. Dubovikov, as part of a larger investigation into turbulence modeling, have also presented results for turbulence subjected to slowly varying, non-self-similar forcing (unpublished). These authors also find an expression for a k-dependent eddy viscosity, though it is not equivalent to that found here due to the essentially different form adopted for the random-force correlation.

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